

Compact maximal hypersurfaces in stably causal spacetimes

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Abstract

Several uniqueness results on compact maximal hypersurfaces in a wide class of stably causal spacetimes are given. They are obtained from the study of a distinguished function on the maximal hypersurface, under suitable natural first order conditions of the spacetime. As a consequence several applications to Geometric Analysis are given.

Keywords: Compact hypersurface, maximal hypersurface, stably causal spacetime.

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1 Introduction

A continuous function on a spacetime, which is strictly increasing on any future directed causal curve is called a time function. In 1968, Hawking proved that any stably causal spacetime admits a time function [15]. Conversely, Hawking also proved that a spacetime is stably causal if it admits a smooth function with a everywhere timelike gradient. More recently, Bernal and Sanchez in [4] define a temporal function as a smooth function on the spacetime with (past-directed) timelike gradient on all the spacetime and prove that any spacetime which admits a time function also admits a temporal function. Thus, as the authors highlight, this result combined with Hawking's ones, ensures, on one hand, that any stable causal spacetime admits a temporal function and, on the other, that any spacetime which admits a time function is stably causal.

On the other hand, there exist a serie of folk questions refered to differentiability of time functions and Cauchy hypersurfaces and smooth orthogonal splitting of globally hyperbolic spacetimes. The principal Theorem in [4] assures that an $(n + 1)$ -dimensional globally hyperbolic spacetime M is isometric to the smooth product manifold

$$\mathbb{R} \times \mathcal{S}, \quad \bar{g} = -\beta d\mathcal{T}^2 + \hat{g} \quad (1)$$

where \mathcal{S} is a smooth spacelike Cauchy hypersurface, $\mathcal{T} : \mathbb{R} \times \mathcal{S} \rightarrow \mathbb{R}$ is the natural projection, $\beta : \mathbb{R} \times \mathcal{S} \rightarrow (0, \infty)$ a smooth function and \hat{g} a 2-covariant symmetric tensor field on $\mathbb{R} \times \mathcal{S}$, satisfying:

- i) $\nabla \mathcal{T}$ is timelike and past-pointing on all M .

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ii) Each hypersurface $\mathcal{S}_\mathcal{T}$ at constant \mathcal{T} is a Cauchy hypersurface, and the restriction $\hat{g}_\mathcal{T}$ of \hat{g} to such a $\mathcal{S}_\mathcal{T}$ is a Riemannian metric.

iii) The radical of \hat{g} at each $w \in \mathbb{R} \times \mathcal{S}$ is $\text{Span} \nabla \mathcal{T}$ at w .

If we denote $\pi_\mathcal{S}$ the canonical projection on the second factor, which is a diffeomorphism restricted to each level hypersurface $\mathcal{S}_\mathcal{T}$, via $\pi_\mathcal{S}$, we can give an one-parameter family of Riemannian metric $g_\mathcal{T}$ on the differentiable manifold \mathcal{S} .

Inspired in all these facts, we consider a natural family of stably causal spacetimes which allows certain accuracy to the study of maximal hypersurfaces.

Consider a differentiable manifold F , an open interval $I \subset \mathbb{R}$ and an one-parameter family $\{g_t\}_{t \in I}$ of Riemannian metric on F , i.e., a smooth map $\Lambda : I \times F \rightarrow T_{0,2}(F)$, where $T_{0,2}(F)$ denote the fiber bundle of 2-covariant tensor on F , such that $\Lambda_t : F \rightarrow T_{0,2}(F)$ is a positive definite metric tensor for all $t \in I$. The product manifold $M = I \times F$ can be endowed with the Lorentzian metric given at each point $(t, p) \in I \times F$ by

$$\bar{g} = \beta \pi_I^*(-dt^2) + \pi_F^*(g_t) \quad (\bar{g} = -\beta dt^2 + g_t \text{ in short}),$$

where π_I and π_F denote the canonical projection onto I and F respectively, and $\beta \in C^\infty(I \times F)$ is a positive function.

We will say that a spacetime is *orthogonal-splitted*, if it is isometric to a spacetime $(M, \bar{g} = -\beta dt^2 + g_t)$. Observe that every orthogonal-splitted spacetime is stably causal.

On the other hand, maximal hypersurfaces play a relevant role in General Relativity and Lorentzian Geometry by several reasons. Among them, it should be emphasized that this class of hypersurfaces has an important role in the analysis of the Cauchy problem with the purpose of dealing with simpler constraint equations or solve them, [10], [11], [12, Chap. VI]. Also, in the proof of positivity of gravitational mass, maximal hypersurfaces appear as useful geometric objects (see [26]). On the other hand, these hypersurfaces admit another interpretation: they describe the transition between expanding and contracting phases of the universe, in some relevant cases [5].

A general summary of several reasons justifying the importance of maximal (and constant mean curvature) hypersurfaces in General Relativity can be found in [18].

From a mathematical point of view, the maximal hypersurfaces arise naturally as critical points of the n -dimensional area functional for compactly supported normal variations. A global and motivating result concerning maximal hypersurfaces is the Calabi-Bernstein theorem. This theorem asserts that the only complete maximal hypersurfaces in the Lorentz-Minkowski spacetime, \mathbb{L}^n , are the spacelike hyperplanes. This fact was proved by Calabi [7] for $n \leq 4$, and extended to arbitrary dimension by Cheng and Yau [8]. In another spacetimes, the problem of characterizing maximal hypersurfaces has become an interesting and amusing research topic, and its study can be considered classic. For instance, in [6], Brill and Flaherty replaced Minkowski spacetime with a spatial closed universe, and proved uniqueness at large by assuming $\text{Ric}(z, z) > 0$ for all timelike vectors z (ubiquitous energy condition). In [18], this energy condition was relaxed by Marsden and Tipler to include, for instance, non-flat vacuum spacetimes. On the other hand, Bartnik, [3], proved very general existence theorems and consequently, he claimed that it would be useful to find new satisfactory uniqueness results. More recently, in [2], Alías, Romero and Sánchez proved new uniqueness results in the class of spatially closed generalized Robertson-Walker spacetimes (which include the spatially closed Robertson-Walker spacetimes), under the Temporal Convergence Condition.

Moreover, in [1], Alías and Montiel improve some of these results and, employing a generalized maximum principle due to Omori [19] and Yau [28], give a general uniqueness result for the case of complete constant mean curvature spacelike hypersurfaces under certain boundedness of the Ricci curvature. Finally, in [22], Romero, Rubio and Salamanca give uniqueness results in the maximal case, for spatially parabolic generalized Robertson-Walker spacetimes, which are open models, whose fiber must have a parabolic universal Riemannian covering (see [22] and [23]).

The principal aim of this work is to study uniqueness of compact maximal hypersurfaces in orthogonal-splitted spacetimes which satisfy certain natural first order conditions, i.e., avoiding curvature assumptions. A characterization of orthogonal-splitted spacetimes which admits a compact spacelike hypersurface (i.e., the spacetime is spatially closed) is provided (Prop. 1). Then, we will prove that, under certain hypothesis, the compact maximal hypersurfaces must be level hypersurfaces of the time function.

In Section 3, we assume certain homogeneity in the expansive or contractive behaviour of the spacetime. More specifically, for each $v \in T_q F$, $q \in F$, denote by \tilde{v} the lift of v on the integral curve $\alpha_q(s) = (s, q) \in I \times F$. We say that the spacetime (M, \bar{g}) is *non-contracting in all directions* if $\partial_t \beta \leq 0$ and, $\partial_t \bar{g}(\tilde{v}, \tilde{v}) \geq 0$, for all $q \in F$ and every $v \in T_q F$. Analogously, we say that the spacetime is *non-expanding in all directions* if the previous inequalities are reversed. Observe that the spacetime is non-contracting (resp. non-expanding) if and only if $\mathcal{L}_{\partial_t} \bar{g}$ semi-definite positive (resp. semi-definite negative), where \mathcal{L} denotes the Lie derivative.

If we consider the observer field $U = \frac{1}{\sqrt{\beta}} \partial_t$, the proper time τ of the observers in U is given by $d\tau = \sqrt{\beta} dt$. As a consequence, the assumption $\partial_t \beta \leq 0$ (resp. $\partial_t \beta \geq 0$) assures that the rate of change (acceleration) of the proper time respect to the time function t is non-increasing (resp. non-decreasing). Observe also that the Lorentzian length $|\partial_t| = -\sqrt{\beta}$ is non-decreasing (resp. non-increasing) along to the integral curves of ∂_t . Observe that all physical interpretations here can be considered locally.

On the other hand, the spacelike assumption $\partial_t \bar{g}(\tilde{v}, \tilde{v}) \geq 0$ (resp. $\partial_t \bar{g}(\tilde{v}, \tilde{v}) \leq 0$), guarantees that an observer in U measures non-contraction (resp. non-expansion) in all directions of its physical space.

There are many important examples of orthogonal-splitted spacetimes satisfying these conditions. For instance, any standard static spacetime $M = I \times F$, $\bar{g} = -\beta dt^2 + g$, being $\beta : F \rightarrow (0, \infty)$ a smooth function, obeys clearly our assumptions.

Several generalized Robertson-Walker spacetimes can lie also under this category. Recall that these spacetimes, introduced in [2], are given by the product manifold of an interval of the real line, I and a Riemannian manifold (F, g_F) furnished with Lorentzian metric $\bar{g} = -dt^2 + f^2 g_F$, where $f : I \rightarrow (0, \infty)$ is a smooth function. Thus, a GRW spacetime also satisfies the imposed conditions whenever the derivative f' is signed.

As a last example, let us consider the differentiable manifold $M = I \times F$ endowed with Lorentzian metric $\bar{g} = -\beta(t) dt^2 + f(t) g_F$, where g_F is a Riemannian metric on F . We can rewrite this metric as $\bar{g} = -d\tau^2 + f(t(\tau)) g_F$. If we assume $f'(t) \geq 0$, $f''(t) \geq 0$ and $\beta'(t) \leq 0$, the spacetime M favors to model an accelerated expanding universe.

Our first uniqueness result is enunciated as follow (see section 3).

Let (M^n, \bar{g}) , $n \geq 4$ be an orthogonal-splitted spacetime. Suppose that (M, \bar{g}) is non-contracting (or non-expanding) in all directions. Then each compact maximal hypersurface in M must be a leaf of the orthogonal foliation to vector field ∂_t , that

is, a level spacelike hypersurface of the temporal function t .

Some nice consequences are analyzed, and several applications to relevant class of spacetimes are provided.

In Section 4, the spacetime has a change in its expansive (or contractive) behaviour. In fact, we will assume that there exists an instant t_0 of the temporal function such that the region of the spacetime $t < t_0$ is non-contracting in all directions, while the region $t > t_0$ is non-expanding in all directions. Then, the level hypersurface $t = t_0$ will be called a *level transition spacelike hypersurface*. The second main result is presented,

Let (M^n, \bar{g}) , $n \geq 4$, be an orthogonal-splitted spacetime which has a level spacelike transition hypersurface. Then every compact maximal hypersurfaces in M must be a level spacelike hypersurface.

Finally, the last section is devoted to present some applications to Geometric Analysis.

2 Preliminaries

Let (M, \bar{g}) be an orthogonal-splitted spacetime. An immersion of a connected n -dimensional manifold $x : S \rightarrow M$ is said to be *spacelike* if the induced metric g on S is Riemannian. In this case, we will refer to S as a spacelike hypersurface. For each spacelike hypersurface S in M we can take $N \in \mathfrak{X}^\perp(S)$ as the only globally unitary timelike vector field normal to S in the same time-orientation of the vector field $-\partial_t$ (i.e., such that $\bar{g}(N, -\partial_t) < 0$). From the Lorentzian metric, $\bar{g}(N, \partial_t) = -\sqrt{\beta} \cosh \theta$, where θ is the *hyperbolic angle* between S and $-\partial_t$ at any point of S (see, for instance, [20, Prop. 5.30]). Denoting π_t the projection onto the time-coordinate factor, for any spacelike hypersurface $x : S \rightarrow M$, we define the function $\tilde{t} = \pi_t \circ x$ on S .

Along this paper, the leaves of the orthogonal foliation to the vector field ∂_t are also called level spacelike hypersurfaces. Note that a spacelike hypersurface S is contained in a level spacelike hypersurface if and only if the function time t is constant on S .

Let $\partial_t^\top := \partial_t + \bar{g}(N, \partial_t)N$ be the tangential component of ∂_t along S . It is easy to see that

$$\nabla \tilde{t} = -\frac{1}{\beta} \partial_t^\top, \quad (2)$$

where ∇ denotes here the gradient on S .

From the Gauss and Weingarten formulas we have

$$\bar{\nabla}_X Y = \nabla_X Y - g(AX, Y)N, \quad (3)$$

for all $X, Y \in \mathfrak{X}(S)$, where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of S and M , and A is the shape operator associated to N ,

$$AX := -\bar{\nabla}_X N.$$

Recall that the *mean curvature function* relative to N is given by $H := -(1/n)\text{tr}(A)$. A spacelike hypersurface is called *maximal* provided that $H = 0$.

On the other hand, a maximal hypersurface is a critical of the volume variation of certain family of hypersurfaces (see [17]). Indeed, let $x : S \longrightarrow M$ an isometric compact immersed spacelike hypersurface into the spacetime. We shall speak indistinctly of S and its image if no confusion arise. Let ω_s be the canonical volume element of S with the Riemannian metric induced from (M, \bar{g}) . The n -volume of S is given by

$$V_S = \int_S \omega_s.$$

We wish know the variation of V_S and ω_s , when we perform a deformation of the hypersurface S . Let ξ be a differentiable vector field defined on a neighbourhood of $x(S)$. Then, ξ generates a local transformation one-parameter group $\{\varphi_s\}_{s \in J}$ of local transformations, where s is the canonical parameter and $J \subset \mathbb{R}$ an open interval containing $s = 0$. We can define a one-parameter family of isometric immersions of hypersurfaces $x : S \longrightarrow M$ given by $x_s = \varphi_s \circ x$. Observe that $x_0 = x$. The corresponding canonical n -volume element for each x_s is noted by ω_s ($\omega_0 \equiv \omega_s$). Then a standart calculation leads to

$$\frac{dV_{x_s}}{ds}(0) = \int_S (\overline{\text{div}} \xi + \bar{g}(\xi, \vec{H}) \omega_0).$$

If S is maximal, the Gauss theorem shows that $\frac{dV_{x_s}}{ds}(0) = 0$.

3 Spacetimes with homogeneity in its expanding-contracting behaviour

First of all, we study conditions under which an orthogonal splitted spacetime, $(M = I \times F, \bar{g})$ admits a compact spacelike hypersurface. Observe that if F , is compact, then the spacetime is spatially closed. Moreover, it can be proved the converse, as the following reasoning shows. Let S be a (connected) compact hypersurface in the spacetime M and $[t_0, t^0] \subset I$ be a closed interval such that $\pi_I(S) = [t_0, t^0]$. Consider the canonical projection projection

$$\pi_F : S \longrightarrow F.$$

Let TU be the fibre bundle of unitary tangent vector on S , for all $u \in TU$

$$g(u, u) \leq g_{\pi_I(u)}(d\pi_F(u), d\pi_F(u)) = A(t, u)g_{t_0}(d\pi_F(u), d\pi_F(u)).$$

Thus, the smooth function $A(t, u)$ is defined on the compact $[t_0, t^0] \times TU$. Let \bar{A} be the maximum of $A(t, u)$. so, we have

$$g(v, v) \leq \bar{A}g_{t_0}(d\pi_F(v), d\pi_F(v)),$$

Hence, up a positive constant factor, it is clear that $d\pi_F$ increases the norms of the tangent vector on (F, g_{t_0}) . Then, making use of [13, Lemma 3.3, Chap. 7], we obtain that π_F is a covering map.

Proposition 1 *i) If an orthogonal-splitted spacetime, $(I \times F, \bar{g})$ admits a compact spacelike hypersurface, then F is compact.*

ii) Moreover, if the universal Riemannian covering of F is compact and S is a complete spacelike hypersurface such that, the function $A(t, u)$, with $t \in \pi_I(S)$ and $u \in TU$ is bounded, then S is compact.

Previous result extends Prop. 3.2 in [2] properly to a wider family of spacetimes.

In order to obtain our results, we develop a formula which will be key. This formula is based on the computation of the Laplacian of the function \tilde{t} on a maximal hypersurface S immersed in a orthogonal-splitted spacetime (M, \bar{g}) .

Let (S, g) be a maximal hypersurface immersed in the spacetime M , where g denotes the induced Riemannian metric on S . Taking into account (2), a straightforward computation leads to

$$\Delta \tilde{t} = \frac{1}{\beta^2} \partial_t^\top(\beta) - \frac{1}{\beta} \operatorname{div}(\partial_t^\top), \quad (4)$$

where Δ and div denote the Laplacian and divergence operators of the immersion S . Now, we can write

$$\operatorname{div}(\partial_t^\top) = \overline{\operatorname{div}}(\partial_t) + \overline{\operatorname{div}}(\bar{g}(N, \partial_t)N) + \bar{g}(\bar{\nabla}_N \partial_t^\top, N),$$

where $\overline{\operatorname{div}}$ denote the divergence operator of the spacetime. Since the hypersurface is maximal, we obtain

$$\operatorname{div}(\partial_t^\top) = \overline{\operatorname{div}}(\partial_t) + \bar{g}(\bar{\nabla}_N \partial_t, N). \quad (5)$$

On the other hand, the term $\overline{\operatorname{div}}(\partial_t)$ has a geometrical interpretation and, in fact, it can be written in terms of certain volume function. Let Ω be the canonical Lorentzian volume element (see, for instance, [20]) of the spacetime. Making use of the Cartan's theorem,

$$d \circ i_{\partial_t} \Omega = \overline{\operatorname{div}}(\partial_t) \Omega.$$

Consider the level spacelike hypersurface $\{t = t_0\}$, $t_0 \in I$, and let ω_0 be the canonical Riemannian volume element on (F, g_{t_0}) . Since the spacetime has an orthogonal splitting, then $\Omega = \sqrt{\beta} dt \wedge \operatorname{vol}_{\text{slice}} \omega_0$, where the function $\operatorname{vol}_{\text{slice}}$ satisfies that $\operatorname{vol}_{\text{slice}} \omega_0 = \omega_t$, being ω_t the canonical Riemannian volume element of (F, g_t) . Hence, we have

$$\overline{\operatorname{div}}(\partial_t) = \partial_t \log \operatorname{vol}_{\text{slice}} + \frac{1}{2} \frac{\partial_t \beta}{\beta}. \quad (6)$$

On the other hand, let us write the normal vector as follows,

$$N = -\frac{\cosh \theta}{\sqrt{\beta}} \partial_t + N^\perp,$$

where $\bar{g}(N^\perp, \partial_t) = 0$. Hence,

$$\bar{g}(\bar{\nabla}_N \partial_t, \partial_t) = \frac{\cosh^2 \theta}{\beta} \bar{g}(\bar{\nabla}_{\partial_t} \partial_t, \partial_t) + \frac{\cosh \theta}{\beta} \{ \bar{g}(\bar{\nabla}_{\partial_t} \partial_t, N^\perp) + \bar{g}(\bar{\nabla}_{N^\perp} \partial_t, \partial_t) \} + \bar{g}(\bar{\nabla}_{N^\perp} \partial_t, N^\perp). \quad (7)$$

Let $q \in S$ be such that $N^\perp(p) \neq 0$. Take coordinates $(\mathcal{U}, (t \equiv x_0, x_1, \dots, x_n))$ around p in the product manifold $I \times F$, such that $N^\perp = \partial_{x_1}$ on $S \cap \mathcal{U}$. Therefore, taking into account the equality which determines the Christoffel symbols for a coordinate system, we have

$$\bar{g}(\bar{\nabla}_{\partial_t} \partial_t, N^\perp) = -\frac{1}{2} \partial_{x_1} \bar{g}(\partial_t, \partial_t)$$

and

$$\bar{g}(\bar{\nabla}_{N^\perp} \partial_t, \partial_t) = \frac{1}{2} \partial_{x_1} \bar{g}(\partial_t, \partial_t).$$

Furthermore,

$$\bar{g}(\bar{\nabla}_{N^\perp} \partial_t, N^\perp) = \frac{1}{2} \partial_t \bar{g}(\partial_{x_1}, \partial_{x_1}).$$

Now, using (5) and (6) we get

$$\bar{g}(\bar{\nabla}_N \partial_t, N) = -\frac{\cosh^2 \theta}{2\beta} \partial_t \beta + \frac{1}{2} \partial_t \bar{g}(\partial_{x_1}, \partial_{x_1}).$$

Finally from (4) we arrive to

$$\Delta \tilde{t} = \frac{1}{\beta^2} \partial_t^\top \beta - \frac{1}{\beta} \left[\partial_t \log \text{vol}_{\text{slice}} - \frac{1}{2} \frac{\sinh^2 \theta}{\beta} \partial_t \beta + \frac{1}{2} \partial_t \bar{g}(\partial_{x_1}, \partial_{x_1}) \right]. \quad (8)$$

Now, assuming that the spacetime has dimension greater or equal than four, we can consider, on S , the pointwise conformal metric to the induced one $\tilde{g} = \beta^{2/(n-2)} g$. In terms of this new metric, equation (8) changes to

$$\tilde{\Delta} \tilde{t} = -\beta^{-n/(n-2)} \left[\partial_t \log \text{vol}_{\text{slice}} - \frac{1}{2} \frac{\sinh^2 \theta}{\beta} \partial_t \beta + \frac{1}{2} \partial_t \bar{g}(\partial_{x_1}, \partial_{x_1}) \right]. \quad (9)$$

Observe that the term $\partial_t \log \text{vol}_{\text{slice}}$ can be interpreted physically as the increasing or decreasing of the local volume of the physical space for the observer in U .

Therefore, if the spacetime is non-contracting or non-expanding in all directions, we have for every point $q \in S$ such that $N^\perp(q) \neq 0$, $\tilde{\Delta} \tilde{t}$ is equally signed.

On the other hand, let $q' \in S$ be with $N^\perp(q') = 0$ and assume that there is no an open set $\mathcal{V} \in S$, with $q' \in \mathcal{V}$ such that $N^\perp(p) = 0$ for all $p \in \mathcal{V}$. As a consequence, we can take a sequence $\{q_n\}_{n=1}^\infty$ in \mathcal{V} such that $\lim_{n \rightarrow \infty} q_n = q'$, $N^\perp(q_n) \neq 0$ for all n and $\tilde{\Delta} \tilde{t}(q_n)$ has equal sign for all n . Thus by continuity $\tilde{\Delta} \tilde{t}(q')$ is equally signed. Finally, if there exists an open set $\mathcal{V} \subset S$ such that $N^\perp(p) = 0$ for all $p \in \mathcal{V}$, then $\tilde{\Delta} \tilde{t} = 0$ on \mathcal{V} .

We are now in position to state the following result,

Theorem 2 *Let (M^n, \bar{g}) , $n \geq 4$, be an orthogonal-splitted spacetime. Assume that (M, \bar{g}) is non-contracting or non-expanding in all directions. Then every compact maximal hypersurfaces in M must be a level spacelike hypersurface.*

Proof. From the assumptions, equation (9) reads as \tilde{t} is a superharmonic or subharmonic function on a compact Riemannian manifold. Then, \tilde{t} must be constant. \square

Remark 3 a) As equation (9) shows, if such hypersurface $\{t = t_0\}$ exists, then it must be satisfied $\partial_t \beta(t_0) = 0$, $\partial_t \log \text{vol}_{\text{slice}}(t_0) = 0$. **b)** A simply calculation shows that

$$\frac{d\omega_s}{ds}(0) = \frac{1}{2} \text{trace}_s [x^*(\mathcal{L}_\xi \bar{g})] \omega_0.$$

Assuming that the level hypersurface $\{t = t_0\}$ is maximal and taking $\xi = \partial_t$, we obtain that

$$\frac{1}{2} \text{trace}_s [x^*(\mathcal{L}_{\partial_t} \bar{g})] = 0.$$

If moreover, ∂_t is a conformal vector field, then $\rho|_{t=t_0} = 0$, when ρ denotes the conformal factor.

As the proof of previous result shows, when $\beta \equiv 1$, then the conformal change does not apply and, consequently, we can also study lower dimensions spacetimes.

Corollary 4 *Let $(I \times F, -dt^2 + g_t)$ be an orthogonal-splitted spacetime, with $\beta \equiv 1$, such that the physical space of the observer in ∂_t non-expands (resp. non-contracts) in all directions. Then the compact maximal hypersurfaces must be a level spacelike hypersurfaces.*

Remark 5 Note that when $\beta \equiv 1$ the vector field ∂_t is a observer field, with proper time t and such that the observers in ∂_t are proper time synchronizables. Moreover, each level spacelike hypersurface $F(t) := \{t\} \times F$ at t constant is a restspace of ∂_t , being each event in $F(t)$ simultaneous for all the observer in ∂_t .

Let $p, q \in F$ be, the distance between $p_t := (t, p)$ and $q_t := (t, q)$ measures by a observer at the instant t is given by $d_t(p_t, q_t)$, where d_t is the riemannian distance in $F(t)$. So, the spacetime is non-contracting (resp. non-expanding) in all directions if and only if $d_t(p_t, q_t)$ is non-decreasing (resp. non-increasing) for all $p, q \in F$.

Now, we can provide some consequences of previous results for certain relevant spacetimes. The first application that we can consider is relative to multiply warped product spacetimes. Consider an interval of the real line I , n Riemannian manifolds (F_i, g_i) and n positive functions $f_i \in C^\infty(I)$ in order to endow the product $I \times F_1 \times \dots \times F_n$ with the Lorentzian metric $-dt^2 + f_1^2 g_1 + \dots + f_n^2 g_n$.

Multiply warped products are interesting in Cosmology. For instance, it is well-known that the Kasner spacetime can be regarded as a multiply warped product [14]. Also, in the study of $(2+1)$ BTZ Black Hole, the multiply warped products are considered [16].

Observe that, if $f_i = f$ for all i , then the spacetime is a generalized Robertson-Walker spacetime.

Corollary 6 *Let $(I \times F_1 \times \dots \times F_m, -dt^2 + \sum_{i=1}^m f_i^2(t)g_{F_i})$ be a multiply warped product spacetime, such that, for all i , the functions $f_i'(t)$ have a common sign. Then every compact maximal hypersurface in M must be a level spacelike hypersurface.*

In particular, in every Lorentzian product, $(\mathbb{R} \times F, -dt^2 + g_F)$ the only compact maximal hypersurfaces are the spacelike level hypersurfaces.

It can be also treated the case of standard static spacetimes (compare with [25, Prop. 2]),

Corollary 7 *In an $n(n \geq 4)$ -dimensional standard static spacetime $(I \times F, -h^2 dt^2 + g_F)$, the only compact maximal hypersurfaces are the level spacelike hypersurfaces.*

Observe that, in the previous case, every level spacelike hypersurface $\{t = t_0\}$ is totally geodesic.

Finally, we can also consider the case of certain Lorentzian twisted products. Recall that the product manifold of an open interval I of the real line, and a Riemannian manifold (F, g_F) , endowed with the Lorentzian metric $\bar{g} = -dt^2 + \lambda g_F$, where $\lambda \in C^\infty(I \times F)$ is a positive function, is a Lorentzian twisted product. In this case, it is well-known that the level hypersurfaces at t constant are totally umbilic (see [21]).

Corollary 8 *In a Lorentzian twisted product spacetime $(I \times F, -dt^2 + \lambda g_F)$, such that $\partial_t \lambda \geq 0$ (resp. ≤ 0) Then every compact maximal hypersurfaces in M is a level spacelike hypersurfaces at t constant.*

4 Behaviour of the volume function

In this section, we consider spacetimes whose expanding or contracting behaviour changes its monotonicity. For these purposes, we introduce the following notion. We say that in an orthogonal-spplitted spacetime, $(I \times F, -\beta dt^2 + g_t)$, a level spacelike hypersurface $t = t_0$ is a *level transition hypersurface* if the spacetime is non-contracting in the region $t \leq t_0$ and non-expanding in the region $t \geq t_0$. Note that if there exist two different spacelike level transition hypersurfaces, then the portion of the spacetime contained between them has geometrical and physical meaning. In geometrical terms, in this region the -metric of the- spacetime has no dependance with the t -coordinate; in physical terms, the spacetime does not evolve here (it is standard static in this region).

This change of behaviour is similar than the spatially closed Friedmann model present (see, for instance, [20, Ch. 12]).

It is easy to show that a level spacelike hypersurface has mean curvature $nH = \frac{1}{\sqrt{\beta}} \overline{\text{div}}(\partial_t) - \frac{1}{2\beta^{3/2}} \partial_t \beta$. Then, any level transition hypersurface must be maximal.

Our second main result,

Theorem 9 *Let (M^n, \overline{g}) , $n \geq 4$, be an orthogonal-splitted spacetime which has a level spacelike transition hypersurface. Then every compact maximal hypersurfaces in M must be also a level spacelike hypersurface.*

Proof. Let $\{t = t_0\}$ be the level spacelike transition hypersurface. Consider the function $\tilde{t} - t_0$ in a compact maximal hypersurface. From (9), it is found that

$$(\tilde{t} - t_0) \tilde{\Delta}(\tilde{t} - t_0) \geq 0.$$

This fact implies that $\tilde{t} - t_0$ is constant, and so the hypersurface must be a level spacelike hypersurface. To see clear this last step, consider any function $h \in C^\infty(P)$, where P is a compact Riemannian manifold, such that $h\Delta h \geq 0$. Then, taking into account Δh^2 and the Divergence Theorem, it is no difficult to see that h must be constant. This applied to the function of this proof, ending it. \square

The previous theorem can be applied to a wide family of spacetimes. For instance, let (F_i, g_i) , $1 \leq i \leq m$, be a finite set of compact Riemannian manifolds. Consider the Lorentzian multiply warped product spacetime $(\mathbb{R} \times F_1 \times \dots \times F_m, -dt^2 + \sum_i a_i^2 \exp(-b_i^2 t^2) g_i)$, where a_i, b_i are not zero constants. As a consequence of the Theorem 9, in this spacetime the only compact maximal hypersurface is $\{t = 0\}$.

Remark 10 The concept of transition hypersurface can be given, in a similar way, for a spacelike hypersurface (non-necessarily for a spacelike level hypersurface). Nevertheless a such transition hypersurface may not be maximal, as the following example shows. Consider the spacetime $(I \times \mathbb{S}^1, -dt^2 + (-t^2 + 2t \sin x + 3)dx^2)$, where $I \subset \mathbb{R}$ is the maximal domain of the t coordinate where the spacetime can be defined. Here, the transition hypersurface is given by $t = \sin x$.

Remark 11 If the inequalities are reversed in the definition of transition hypersurface, counterexamples to uniqueness results can be shown. For example, consider the de Sitter spacetime, $(\mathbb{R} \times \mathbb{S}^n, -dt^2 + \cosh^2(t)g_{\mathbb{S}^n})$. Every rigid motion of the level spacelike hypersurface $\{t = 0\}$ is a maximal hypersurface. Observe that, in this case, $\{t = 0\}$ constitutes the level spacelike transition hypersurface with reversed behaviour.

Again, when $\beta \equiv 1$, it can be also treated the case of dimension $2 + 1$.

Corollary 12 *Let $(I \times F, -dt^2 + g_t)$ be an orthogonal-splitted spacetime which has a level spacelike transition hypersurface. Then every compact maximal hypersurface in M must be a level spacelike hipersurface.*

5 Applications to Geometric Analysis

We will begin this section showing that every compact spacelike hypersurface in an orthogonal-splitted spacetime is a maximal hypersurface in a suitable pointwise conformal spacetime. Next, our uniqueness result are applied to study of several Geometric Analysis's problems, which arise naturally from Lorentzian Geometry.

Let S be a spacelike hypersurface isometrically immersed in an othogonal-splitted spacetime (M, \bar{g}) . It can be considered the hypersurface S isometrically immersed in the pointwise conformal spacetime $(M, \tilde{g} = e^{2\alpha}\bar{g})$, where $\alpha \in C^\infty(I \times F)$. The unitary normal vector field N of S in (M, \bar{g}) , is related with the unitary normal vector field \tilde{N} of S in $(M, \tilde{g} = e^{2\alpha}\bar{g})$ by $\tilde{N} = e^{-\alpha}N$. Observe that if S is spacelike in (M, \bar{g}) , then S is also spacelike in (M, \tilde{g}) . If we take $\{E_i\}_{i=1}^n$ a local frame field on S in (M, \bar{g}) , then $\{e^{-\alpha}E_i\}_{i=1}^n$, denoted by $\{\tilde{E}_i\}_{i=1}^n$, is a local frame field on S in (M, \tilde{g}) . Denoting by H and \tilde{H} the mean curvature of S in (M, \bar{g}) and (M, \tilde{g}) , respectively, we have

$$n\tilde{H} = \sum_{i=1}^n \tilde{g}(\tilde{\nabla}_{\tilde{E}_i} \tilde{N}, \tilde{E}_i) = \sum_{i=1}^n g(\tilde{\nabla}_{E_i} e^{-\alpha} N, E_i) = \sum_{i=1}^n e^{-\alpha} g(\tilde{\nabla}_{E_i} N, E_i),$$

where $\tilde{\nabla}$ is the Levi-Civita connection of \tilde{g} . Then from the previous equation it follows

$$e^\alpha \tilde{H} = H + \bar{g}(\bar{\nabla} \alpha, N). \quad (10)$$

On the other hand, we need the following technical result,

Lemma 13 *Let S be a compact spacelike hypersurface immersed in orthogonal-splitted spacetime (M, \bar{g}) and N its unitary normal vector field. Then, for every function $h \in C^\infty(S)$, there exists a function $\bar{\alpha} \in C^\infty(M)$ such that*

$$(\nabla \bar{\alpha})|_S^\perp = -hN. \quad (11)$$

Proof. The compactness of S allows to take an open interval J such that the geodesics $\gamma_p(s)$ starting at $p \in S$ satisfying $\gamma_p'(s) = N(p)$ are defined for all $p \in S$ and for all $s \in J$. On the other hand, the function h can be extended on a tubular neighbourhood $\mathcal{U} = \{\gamma_p(s)/t \in J, p \in S\}$ of S , being \bar{h} constant along each geodesic $\gamma_p(s)$, with value $h(p)$. Moreover, the vector field defined in each point of U by $\gamma_p'(s)$, also extends to the normal vector field N . Obviously, its flow on $I \times S$ is given by $\varphi(t, p) = \gamma_p(t)$, being φ bijective. Consider the

smooth function $\pi_J \circ \varphi^{-1}$. Thus, the normal component gradient $\bar{\nabla} \bar{h}(\pi_J \circ \varphi^{-1})|_S^\perp = hN$, since $\bar{g}(N, \bar{\nabla} h) = 0$.

Now, let ϕ a function on M such that $0 \leq \phi(p) \leq 1$, for all $p \in M$, satisfying (see Corollary in Section 1.11 of [27]),

- i) $\phi(p) = 1$ if $p \in \{\gamma_t(p)/t \in J', p \in S\}$, being $J' \subset J$ an closed interval with $0 \in J'$.
- ii) $\text{supp } \phi \subset U$.

Therefore, the function ϕ allows to extend the function $\pi_J \circ \varphi^{-1}$ on all M and to obtain $\bar{\alpha}$. \square

Lemma 14 *In an orthogonal-splitted spacetime every compact spacelike hypersurface is a maximal hypersurface in a suitable pointwise conformal spacetime.*

Let (M, \bar{g}) be an orthogonal-splitted spacetime and let $\alpha \in C^\infty(F)$ a smooth function. Consider the natural extension $\bar{\alpha} = \alpha \circ \pi_F$ on M . Let S be a compact maximal hypersurface in (M, \bar{g}) , then from (10) the mean curvature function of S isometrically immersed in the conformal spacetime $(M, e^{2\bar{\alpha}}\bar{g})$ is given by

$$\tilde{H} = e^{-\bar{\alpha}}\bar{g}(\bar{\nabla}\bar{\alpha}, N).$$

Note that S is the only compact maximal hypersurface in (M, \bar{g}) if and only if S is the only compact spacelike hypersurface in $\bar{\alpha} = \alpha \circ \pi_F$ with mean curvature $\tilde{H} = e^{-\bar{\alpha}}\bar{g}(\bar{\nabla}\bar{\alpha}, N)$. Making use of this fact, we can obtain some Geometric Analysis's result.

Let F be a compact Riemannian manifold and let $(I \times F, \bar{g})$ be an $n(\geq 4)$ -dimensional orthogonal-splitted spacetime. Let $u : F \rightarrow I$ a smooth function, such that the entire graph $\Sigma_u = \{(u(p), p) : p \in F\}$ is spacelike. Consider $\alpha \in C^\infty(F)$ and denote us by N the unitary normal vector field on Σ_u . If we denote $\tilde{H}(u)$ the mean curvature function on the graph Σ_u isometrically immersed in the conformal spacetime $(I \times F, e^{2\bar{\alpha}}\bar{g})$, with $\bar{\alpha} = \alpha \circ \pi_F$, and we assume that the spacetime $(I \times F, \bar{g})$ is non-contracting or non expanding in all direction, or it has a level spacelike transition hypersurface, then every solution to the equation

$$\tilde{H}(u) = e^{-\bar{\alpha}}g_u(d\pi_F(N), D_u\alpha),$$

being D_u the gradient operator in (F, g_u) , must be constant.

This last can be specialized to the case in which the ambient Lorentzian manifold is a generalized Robertson-Walker spacetime.

Let (F, g_F) be a compact n -dimensional Riemannian manifold and $f : I \rightarrow \mathbb{R}$ a positive smooth function. For each $u \in C^\infty(F)$ such that $u(F) \subseteq I$, we can consider its graph $\Sigma_u = \{(u(p), p) : p \in F\}$ in the Lorentzian warped product $(M = I \times_f F, \bar{g})$. The graph inherits from M a metric, represented on F by

$$g_u = -du^2 + f(u)^2 g_F.$$

This metric is Riemannian (i.e. positive definite) if and only if u satisfies $|Du| < f(u)$ everywhere on F , where Du denotes the gradient of u in (F, g_F) and $|Du|^2 = g_F(Du, Du)$. Note that $\tilde{t}(u(p), p) = u(p)$ for any $p \in F$, and so \tilde{t} and u may be naturally identified on Σ_u .

When Σ_u is spacelike, the unitary normal vector field on Σ_u satisfying $\bar{g}(N, \partial_t) < 0$ is

$$N = \frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} (f(u)^2 \partial_t + Du).$$

Theorem 15 Let (F, g_F) be a compact Riemannian manifold, and the smooth functions $\alpha : F \rightarrow \mathbb{R}$ and $f : I \rightarrow (0, \infty)$, where $I = (a, b)$ is an open interval, satisfying:

- i) f' is signed or
- ii) f is non-decreasing in the interval (a, t_0) and non-increasing in the interval (t_0, b) .

The only entire spacelike graphs $\Sigma_u = \{(u(p), p) : p \in F\}$ in the spacetime $(I \times F, -e^{2\alpha} dt^2 + e^{2\alpha} g_F)$, whose mean curvature \tilde{H} is prescribed by

$$\tilde{H}(u) = \frac{e^{-\alpha}}{f(u)\sqrt{f(u)^2 - |Du|^2}} g_F(D\alpha, Du)$$

must be the constant graph $u = u_0$ such that $f'(u_0) = 0$.

Note that the graph Σ_u is spacelike in $(I \times F, -e^{2\alpha} dt^2 + e^{2\alpha} g_F)$ if and only if it is so in $(I \times F, -dt^2 + f(t)^2 g_F)$. On the other hand, if we conceive the mean curvature function of a graph Σ_u as a known operator $\tilde{H}(u)$ on $C^\infty(F)$, previous theorem can be enunciated in equivalent form,

Theorem 16 Let (F, g_F) be a compact Riemannian manifold, $\alpha : M \rightarrow \mathbb{R}$ a smooth function and $f : I \rightarrow (0, \infty)$ a smooth function, where $I = (a, b)$ is an open interval, satisfying:

- i) f' is signed or
- ii) f is non-decreasing in the interval (a, t_0) and non-increasing in the interval (t_0, b) .

Then, the only solution to the differential equation

$$\tilde{H}(u) = \frac{e^{-\alpha}}{f(u)\sqrt{f(u)^2 - |Du|^2}} g_F(D\alpha, Du),$$

with the restriction $|Du| < f(u)$ are the constant function $u = u_0$, with $f'(u_0) = 0$.

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